Lecture Notes, Lecture 4, 5

1.4 A first approach: existence of general equilibrium in an economy with an excess demand function

N goods

Household's ($h \in H$) initial endowments of goods $r^h = (r^h_1, r^h_2, \ldots, r^h_N) \in R^N$. Aggregate endowment of the economy is $r \equiv \sum_{h \in H} r^h$.

Prices

$p = (p_1, p_2, p_3, \ldots, p_{N-1}, p_N) = (3, 1, 5, \ldots, 0.5, 10)$. Since only relative prices, price ratios, matter in forming demand and supply, we suppose that the price space P, is the unit simplex in R^N.

$$P = \left\{ p \middle| p \in R^N, p_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^{N} p_i = 1 \right\}$$

Household demands, $h \in H$

$D^h : P \to R^N$, $D^h(p) = (D^h{}_1(p), D^h{}_2(p), \ldots, D^h{}_n(p), \ldots, D^h{}_N(p))$

Firm supplies, $j \in F$

$S^j : P \to R^N$ $S^j(p) = (S^j{}_1(p), S^j{}_2(p), \ldots, S^j{}_n(p), \ldots, S^j{}_N(p))$

Excess demand

$$Z(p) = \sum_{h \in H} D^h(p) - \sum_{j \in F} S^j(p) - r \qquad (1.26)$$

$$Z : P \to R^N \qquad (1.27)$$

$$Z(p) \equiv (Z_1(p), Z_2(p), Z_3(p), \ldots, Z_N(p))$$

<u>Assumptions:</u>

Walras' Law : For all $p \in P$,

$$p \cdot Z(p) = \sum_{i=1}^{N} p_i \cdot Z_i(p) = 0 \qquad (1.28)$$

Continuity: Z(p) is a continuous function.

Definition: $p^0 \in P$ is said to be an equilibrium price vector if $Z(p^0) \le 0$ (0 is the zero vector; the inequality applies co-ordinatewise) with $p_i^0 = 0$ for i such that $Z_i(p^0) < 0$. That is, p^0 is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 1.1 (Brouwer Fixed Point Theorem): Let $f(\cdot)$ be a continuous function, $f : P \to P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 1.2: Let Walras' Law and Continuity be fulfilled. Then there is $p^* \in P$ so that p^* is an equilibrium.

Proof: In order to prove the theorem we posit a price adjustment function, T, designed to represent the Walrasian auctioneer, raising prices of goods in excess demand, reducing prices of goods in excess supply, while keeping the price vector on the simplex P. Let $T : P \to P$. We will use the Brouwer fixed point theorem to show that the price adjustment function has a fixed point, a price vector from which it will not further readjust prices. Then we use the Walras Law to show that this fixed point is a market clearing equilibrium. We define T as follows:

$$T(p) = (T_1(p), T_2(p), ..., T_i(p), ..., T_N(p))$$

$$T_i(p) \equiv \frac{\mathrm{Max}[0, p_i + Z_i(p)]}{\sum_{n=1}^{N} \mathrm{Max}[0, p_n + Z_n(p)]} \qquad (1.29)$$

First note that the denominator is the sum over i=1, ..., N, of the numerators. This means that T really is a mapping into the unit simplex. For T to be well defined, the denominator must be nonzero. We state without proof that this will follow from Walras' Law. That is,

$$\sum_{n=1}^{N} \mathrm{Max}[0, p_n + Z_n(p)] \neq 0 \qquad (1.30)$$

Then T is a continuous mapping from P into P. By the Brouwer fixed point theorem there is $p^* \in P$ so that $T(p^*) = p^*$.

This completes the first step of the proof --- showing that the price adjustment process has a stopping point, p*. The next step is to show that p* really is a market-clearing vector of prices. That result depends on how cleverly T(p) is constructed. If T(p) is a well designed price adjustment function in a well behaved economy, then its fixed point, p*, should be a market equilibrium.

Since $T(p^*) = p^*$, for each good k, $T_k(p^*) = p_k^*$. That is, for all k = 1, ..., N,

$$p_k^* = \frac{\mathrm{Max}[0, p_k^* + Z_k(p^*)]}{\sum_{n=1}^{N} \mathrm{Max}[0, p_n^* + Z_n(p^*)]} \qquad (1.31)$$

Either $p_k^* = 0$ (Case 1), or

$$p_k^* = \frac{p_k^* + Z_k(p^*)}{\sum_{n=1}^{N} \mathrm{Max}[0, p_n^* + Z_n(p^*)]} > 0 \quad , \qquad \text{(Case 2).}$$

Case 1: $p_k^* = 0 = \mathrm{Max}[0, p_k^* + Z_k(p^*)]$. Hence $0 \geq p_k^* + Z_k(p^*) = Z_k(p^*)$ and $Z_k(p^*) \leq 0$.

Case 2: To save repeated messy notation define

$$\lambda = \frac{1}{\sum_{n=1}^{N} \mathrm{Max}[0, p_n^* + Z_n(p^*)]} > 0 \qquad (1.34)$$

$$T_k(p^*) = \lambda(p_k^* + Z_k(p^*)) = p_k^* > 0$$

$$(1-\lambda)p_k^* = \lambda Z_k(p^*) \qquad (1.35)$$

multiply through by $Z_k(p^*)$,

$$(1-\lambda)p_k^* Z_k(p^*) = \lambda(Z_k(p^*))^2 \qquad (1.36)$$

sum over all k in case 2,

$$(1-\lambda) \sum_{k\in \text{ Case 2}} p_k^* Z_k(p*) = \lambda \sum_{k\in \text{ Case 2}} (Z_k(p^*))^2 \qquad (1.37)$$

Walras' Law says

$$0 = \sum_{k=1}^{N} p_k^* Z_k(p^*) = \sum_{k\in \text{ Case 1}} p_k^* Z_k(p^*) + \sum_{k\in \text{ Case 2}} p_k^* Z_k(p^*) \qquad (1.38)$$

But for $k \in$ Case 1, $p_k^* Z_k(p^*) = 0$, so

$$0 = \sum_{k\in \text{ Case 1}} p_k^* Z_k(p^*) \ , \qquad (1.39)$$

So

$$\sum_{k\in \text{ Case 2}} p_k^* Z_k(p^*) = 0 \ . \qquad (1.40)$$

Hence from (1.37)[1] we have,

[1] There is a typo in the text at this point. The equation number referred to should be (1.37) as shown here, not "(1.13)" as it appears in the text.

$$0 = (1-\lambda) \sum_{k \in \text{ Case 2}} p_k^* Z_k(p^*) = \lambda \sum_{k \in \text{ Case 2}} (Z_k(p^*))^2 \qquad (1.41)$$

$Z_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in case 2).

Hence, p^* is an equilibrium; it achieves excess demands of zero for all goods with positive prices and prices of zero for all goods in excess supply.

QED